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Order Topology Orthosummability in Quantum Logics

Wu Junde,[1,4] Zhou Su,[2] and Cho Minhyung[3]

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By using the Antosik–Mikusinski infinite matrix convergence theorem in quantum logics, we prove a theorem on orthosummability with respect to order topology in quantum logics.

KEY WORDS: quantum logics; effect algebras; order topologies; orthosummable.

1. EFFECT ALGEBRAS AND ORDER TOPOLOGIES

Let L be a set with two special elements 0, 1, $\perp$ be a subset of $L \times L$, if $(a, b) \in \perp$, write $a \perp b$, and let $\oplus : \perp \to L$ be a binary operation. If the following axioms hold:

(i) Commutative Law: If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
(ii) Associative Law: If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(iii) Orthocomplementation Law: For each $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.
(iv) Zero-Unit Law: If $a \in L$ and $1 \perp a$, then $a = 0$.

Then the algebraic system $(L, \perp, \oplus, 0, 1)$ is said to be an *effect algebra*. This is important for modelling unsharp quantum logics (Foulis and Bennett, 1994).

Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in L$ and $a \perp b$ we say that a and b are *orthogonal*. If $a \oplus b = 1$ we say that b is the *orthocomplement* of a, and we write $b = a'$. Clearly $1' = 0$, $(a'')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$. We say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We may prove

[1] Department of Mathematics, Zhejiang University, Hangzhou, People's Republic of China.
[2] City College, Zhejiang University, Hangzhou, People's Republic of China.
[3] Department of Applied Mathematics, Kumoh National Institute of Technology, Kyungbuk, Korea.
[4] To whom correspondence should be addressed at Department of Mathematics, Zhejiang University, Hongzhou 310027, People's Republic of China; e-mail: wjd@math.zju.edu.cn.

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that $\leq$ is a partial ordering on L and satisfies that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in L$.

Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be a net of $(L, \perp, \oplus, 0, 1)$. Then we write $a_\alpha \uparrow$, when $\alpha \preceq \beta$, $a_\alpha \leq a_\beta$. Moreover, if a is the supremum of $\{a_\alpha : \alpha \in \Lambda\}$, i.e., $a = \vee\{a_\alpha : \alpha \in \Lambda\}$, then we write $a_\alpha \uparrow a$.

Similarly, we may write $a_\alpha \downarrow$ and $a_\alpha \downarrow a$.

If $\{u_\alpha\}_{\alpha \in \Lambda}$, $\{v_\alpha\}_{\alpha \in \Lambda}$ are two nets of $(L, \perp, \oplus, 0, 1)$, for $u \uparrow u_\alpha \leq v_\alpha \downarrow v$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$ and $v_\alpha \downarrow v$. We write $b \leq u_\alpha \uparrow u$ if $b \leq u_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$.

We say a net $\{a_\alpha\}_{\alpha \in \Lambda}$ of $(L, \perp, \oplus, 0, 1)$ is *order convergent* to a point a of L if there exists two nets $\{u_\alpha\}_{\alpha \in \Lambda}$ and $\{v_\alpha\}_{\alpha \in \Lambda}$ of $(L, \perp, \oplus, 0, 1)$ such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$

Let $\mathcal{F} = \{F : F = \emptyset$ or $F \subseteq L$ and for each net $\{a_\alpha\}_{\alpha \in \Lambda}$ of F such that if $\{a_\alpha\}_{\alpha \in \Lambda}$ is order convergent to a, then $a \in F\}$.

It is easy to prove that the family $\mathcal{F}$ of subsets of L defines a topology τ_0^L on $(L, \perp, \oplus, 0, 1)$ such that $\mathcal{F}$ consists of all closed sets of this topology. The topology τ_0^L is called the *order topology* of $(L, \perp, \oplus, 0, 1)$ (Birkhoff, 1948).

If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted by $c = b \ominus a$.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \perp a_2$, $(a_1 \oplus a_2) \perp a_3, \ldots$ and $(a_1 \oplus a_2 \cdots \oplus a_{n-1}) \perp a_n$, we say that F is *orthogonal* and we define $\oplus F = a_1 \oplus a_2 \cdots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ (by the commutative and associative laws, this sum does not depend on any permutation of elements). Now, if A is an arbitrary subset of L and $\mathcal{F}(A)$ is the family of all finite subsets of A, we say that A is *orthogonal* if F is orthogonal for every $F \in \mathcal{F}(A)$. If A is orthogonal, we define $\oplus A = \bigvee\{\oplus F : F \in \mathcal{F}(A)\}$, supposed that the supremum exists in $(L, \leq)$, and it is called the $\oplus$-*sum* of A.

If for all $a, b \in L$, $a \leq b$ or $b \leq a$, then $(L, \perp, \oplus, 0, 1)$ is said to be a *totally ordered effect algebra*; if for all $a, b \in L$, satisfies that $a < b$, there exists $c \in L$ such that $a < c < b$, then $(L, \perp, \oplus, 0, 1)$ is said to be *connect*.

An effect algebra is *complete*, if for each orthogonal subset A of L, the $\oplus$-sum $\oplus A$ exists; if for each countable orthogonal subset B of L, the $\oplus$-sum $\oplus B$ exists, then we say that the effect algebra is σ-*complete*.

2. ORDER TOPOLOGY ORTHOSUMMABILITY

As we know, orthosummability is an important topic in quantum logics (Habil, 1994; Schroder, 1999). In recent, Wu Junde, Lu Shijie, and Kim Dohan studied the $\oplus$-sum and proved a uniform $\oplus$-sum theorem (Junde *et al.*, 2003). In this paper, we introduce the order topology orthosummability of orthogonal sets in effect algebra $(L, \perp, \oplus, 0, 1)$ and prove an order topology orthosummability theorem in $(L, \perp, \oplus, 0, 1)$.

Let $(L, \perp, \oplus, 0, 1)$ be a totally ordered effect algebra. We say that the sequence $\{a_n\}_{n\in\mathbf{N}}$ of $(L, \perp, \oplus, 0, 1)$ is an order topology τ_0^L-*Cauchy sequence*, if for each $h \in L$, $0 < h$, there exists $n_0 \in \mathbf{N}$ such that when $n_0 \leq n$, $n_0 \leq m$, if $a_n \leq a_m$, then $a_m \ominus a_n < h$; if $a_m \leq a_n$, then $a_n \ominus a_m < h$ (Junde *et al.*, 2003).

Definition 1. Let A be an orthogonal subset of $(L, \perp, \oplus, 0, 1)$ and $\mathcal{F}(A)$ be the family of all finite subsets of A. It is clear that $\mathcal{F}(A)$ is a net if we define $F_1 \leq F_2$ iff $F_1 \subseteq F_2$. If the net $\{\oplus F : F \in \mathcal{F}(A)\}$ is order topology τ_0^L convergent to $a \in L$, then we say that A is order topology τ_0^L-*orthosummable* and a is the order topology τ_0^L-*summation* of A.

Lemma 1. (Junde *et al.*, 2003). *Let $(L, \perp, \oplus, 0, 1)$ be a σ-complete totally ordered connect effect algebra. Then for each $h \in L$, $0 < h$, there exists an orthogonal $\oplus$-summable sequence $\{h_i\}$ of L such that $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^{n} h_i\} < h$.*

Furthermore, we can prove the following lemma, it is very important in this paper:

Lemma 2. *Let $(L, \perp, \oplus, 0, 1)$ be a totally order effect algebra, $h = h_1 \oplus h_2$, $g = g_1 \oplus g_2$. If $max\{h, g\} \ominus min\{h, g\} > max\{h_1, g_1\} \ominus min\{h_1, g_1\}$, then $max\{h_2, g_2\} \ominus min\{h_2, g_2\} > (max\{h, g\} \ominus min\{h, g\}) \ominus (max\{h_1, g_1\} \ominus min\{h_1, g_1\})$.*

3. MAIN THEOREM AND ITS PROOF

Now, by using the methods of (Mazario, 2001) and (Aizpuru and Gutierrez-Davila, 2003) and the Antosik-Mikusinski theorem in quantum logics (Junde *et al.*, 2003), we prove the following order topology τ_0^L-orthosummability theorem:

Theorem 1. *Let $(L, \perp, \oplus, 0, 1)$ be a σ-complete totally ordered connect effect algebra, for each $i \in \mathbf{N}$, the orthogonal set $\{a_{i,\alpha}\}_{\alpha\in\Lambda}$ of L be order topology τ_0^L orthosummable, for each finite subset F of Λ, the sequence $\{\oplus_{\alpha\in F} a_{i,\alpha}\}_{i\in\mathbf{N}}$ be order topology τ_0^L convergent, for each pairwise disjoint finite subset sequence $\{E_j\}$ of Λ and each infinite subset D of $\mathbf{N}$, there exist a countable subset B of Λ and an infinite subset M of D such that $E_j \subseteq B$ if $j \in M$ and $E_j \cap B = \emptyset$ if $j \in \mathbf{N}\backslash M$, and $\{\oplus_{\alpha\in B} a_{i,\alpha}\}_{i\in\mathbf{N}}$ be an order topology τ_0^L-Cauchy sequence. Then the orthogonal family $\{a_{i,\alpha}\}_{\alpha\in\Lambda}$ of L is order topology τ_0^L uniformly orthosummable with respect to $i \in \mathbf{N}$.*

Proof: We only need to prove that the nets $\{\oplus_{\alpha\in F} a_{i,\alpha}\}_{F\in\mathcal{F}(\Lambda)}$ are order topology τ_0^L uniformly Cauchy with respect to $i \in \mathbf{N}$. If not, pick a $h \in L$ such that for each $F_0 \in \mathcal{F}(\Lambda)$ there exist $F_0', F_0'' \in \mathcal{F}(\Lambda)$ and $i_0 \in \mathbf{N}$ satisfy $F_0 \subseteq F_0' \subseteq F_0''$, and

$\oplus_{\alpha\in F_0''\setminus F_0'} a_{i_0,\alpha} \geq h$. This shows that for each $F_0 \in \mathcal{F}(\Lambda)$ there exist $F_1 \in \mathcal{F}(\Lambda\setminus F_0)$ and $i_0 \in \mathbf{N}$ such that $\oplus_{\alpha\in F_1} a_{i_0,\alpha} \geq h$. That is,

$$\{\oplus_{\alpha\in F} a_{i_0,\alpha} : F \in \mathcal{F}(\Lambda\setminus F_0)\} \not\subseteq [0, h). \tag{1}$$

We show that (1) will hold for infinite many numbers $i \in \mathbf{N}$. If $\{\oplus_{\alpha\in F} a_{i,\alpha} : F \in \mathcal{F}(A\setminus F_0)\} \not\subseteq [0, h)$ only for $i_1, i_2, \ldots, i_k$, note that for each $i \in \mathbf{N}$, $\{a_{i,\alpha}\}_{\alpha\in\Lambda}$ is order topology τ_0^L-orthosummable, so it follows easily that there exist $F_1, \ldots, F_k \in \mathcal{F}(\Lambda)$ such that

$$\{\oplus_{\alpha\in F} a_{i_j,\alpha} : F \in \mathcal{F}(\Lambda\setminus F_j)\} \subseteq [0, h),\ j = 1, \ldots, k.$$

Let $H = F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_k$. We have

$$\{\oplus_{\alpha\in F} a_{i,\alpha} : F \in \mathcal{F}(\Lambda\setminus H)\} \subseteq [0, h),\ i \in \mathbf{N}.$$

This contradicts (1) and so the conclusion holds. □

This shows that for each $F_0 \in \mathcal{F}(\Lambda)$ and each $i_0 \in \mathbf{N}$, there exist $F \in \mathcal{F}(\Lambda\setminus F_0)$ and $i > i_0$ such that $\oplus_{\alpha\in F} a_{i,\alpha} \geq h$.

Thus, we can obtain a sequence of $\{F_k\}_{k\in\mathbf{N}}$ of pairwise disjoint finite subsets of Λ and an increasing sequence $\{i_k\}_{k\in\mathbf{N}}$ of positive integers such that

$$\oplus_{\alpha\in F_k} a_{i_k,\alpha} \geq h. \tag{2}$$

Let $b_{nk} = \oplus_{\alpha\in F_k} a_{i_n,\alpha}$. Then by the hypothesis of Theorem 1 that b_{nk} satisfies the following conditions:

(i) For each $n \in \mathbf{N}$, $\{b_{nk}\}$ is an orthogonal sequence of L, and $\{b_{nk}\}$ is $\oplus$-summable by the σ-completeness of $(L, \perp, \oplus, 0, 1)$.

(ii) For each finite subset $\mathbf{N_0}$ of $\mathbf{N}$, the sequence $\{\oplus_{k\in\mathbf{N_0}} b_{nk}\}_{n\in\mathbf{N}}$ is order topology τ_0^L convergent.

(iii) For each pairwise disjoint finite subsets sequence $\{B_j\}$ of $\mathbf{N}$ and each infinite subset E of $\mathbf{N}$, there exist a infinite subset G of E and an infinite subset Q of $\mathbf{N}$ such that $B_j \subseteq Q$ if $j \in G$ and $B_j \cap Q = \emptyset$ if $j \in \mathbf{N}\setminus G$, and $\{\oplus_{k\in Q} b_{nk}\}_{n\in\mathbf{N}}$ is an order topology τ_0^L-Cauchy sequence.

Now, we prove that for each $P \subseteq \mathbf{N}$, the sequence $\{\oplus_{k\in P} b_{nk}\}_{n\in\mathbf{N}}$ is order topology τ_0^L Cauchy.

In fact, if not, we can find a $h_1 \in L$ such that for each $n_0 \in \mathbf{N}$, there exist $m, n > n_0$ such that if $\oplus_{k\in P} b_{mk} \geq \oplus_{k\in P} b_{nk}$, then $\oplus_{k\in P} b_{mk} \ominus (\oplus_{k\in P} b_{nk}) \geq h_1$, if $\oplus_{k\in P} b_{nk} \geq \oplus_{k\in P} b_{mk}$, then $\oplus_{k\in P} b_{nk} \ominus (\oplus_{k\in P} b_{mk}) \geq h_1$. It follows from Lemma 1 that there exist three orthogonal elements h_2, h_3, h_4 such that $h_2 \oplus h_3 \oplus h_4 < h_1$, $h_3 \oplus h_4 < h_2$.

Let $n_0 = 1$, $m_1, n_1 > n_0$ and when $\oplus_{k\in P} b_{m_1k} \geq \oplus_{k\in P} b_{n_1k}$, $\oplus_{k\in P} b_{m_1k} \ominus (\oplus_{k\in P} b_{n_1k}) \geq h_1$; when $\oplus_{k\in P} b_{n_1k} \geq \oplus_{k\in P} b_{m_1k}$, $\oplus_{k\in P} b_{n_1k} \ominus (\oplus_{k\in P} b_{m_1k}) \geq h_1$.

It follows from (i) that there exists a $p_1 \in \mathbf{N}$ such that for each $H \subseteq \{p_1 + 1, \ldots, \}$,

$$\oplus_{k\in H} b_{m_1 k} \ominus (\oplus_{k\in H} b_{n_1 k}) \le h_4,$$

or

$$\oplus_{k\in H} b_{n_1 k} \ominus (\oplus_{k\in H} b_{m_1 k}) \le h_4.$$

Thus, it follows from Lemma 2 that

$$\oplus_{k\in P\cap\{1,2,\ldots,\, p_1\}} b_{m_1 k} \ominus (\oplus_{k\in P\cap\{1,2,\ldots,\, p_1\}} b_{n_1 k}) \ge h_2 \oplus h_3,$$

or

$$\oplus_{k\in P\cap\{1,2,\ldots,p_1\}} b_{n_1 k} \ominus (\oplus_{k\in P\cap\{1,2,\ldots,p_1\}} b_{m_1 k}) \ge h_2 \oplus h_3.$$

Note that (ii), there exists $l_1 > m_1, l_1 > n_1$ such that when $m, n > l_1$ and $C \subseteq \{1, 2, \ldots, p_1\}$,

$$\oplus_{k\in C} b_{mk} \ominus (\oplus_{k\in C} b_{nk}) \le h_3,$$

or

$$\oplus_{k\in C} b_{nk} \ominus (\oplus_{k\in P} b_{mk}) \le h_3.$$

Let $n_0 > l_1$ and $m_2, n_2 > n_0$ such that $\oplus_{k\in P} b_{m_2 k} \ominus (\oplus_{k\in P} b_{n_2 k}) \ge h_1$ or $\oplus_{k\in P} b_{n_2 k} \ominus (\oplus_{k\in P} b_{m_2 k}) \ge h_1$.

It follows from (i) again that we can pick a $p_2 \in \mathbf{N}$, $p_2 > p_1$ such that for each $H \subseteq \{p_2 + 1, \ldots, \}$,

$$\oplus_{k\in H} b_{m_2 k} \ominus (\oplus_{k\in H} b_{n_2 k}) \le h_4$$

or

$$\oplus_{k\in H} b_{n_2 k} \ominus (\oplus_{k\in H} b_{m_2 k}) \le h_4.$$

Thus, it follows from Lemma 2 that

$$\oplus_{k\in P\cap\{p_1,\ldots,p_2\}} b_{m_2 k} \ominus (\oplus_{k\in P\cap\{p_1,\ldots,p_2\}} b_{n_2 k}) \ge h_2,$$

or

$$\oplus_{k\in P\cap\{p_1,\ldots,p_2\}} b_{n_2 k} \ominus (\oplus_{k\in P\cap\{p_1,\ldots,p_2\}} b_{m_2 k}) \ge h_2.$$

Inductively, we may obtain three increasing sequences $\{n_i\}$, $\{m_i\}$, and $\{p_i\}$ of $\mathbf{N}$ such that

(iv) When $i > 1$ and $C \subseteq \{1, 2, \ldots, p_{i-1}\}$,

$$\oplus_{k\in C} b_{m_i k} \ominus (\oplus_{k\in C} b_{n_i k}) \le h_3,$$

or

$$\oplus_{k\in C} b_{n_i k} \ominus (\oplus_{k\in C} b_{m_i k}) \le h_3.$$

(v) If $E_i = P \cap \{p_{i-1} + 1, \ldots, p_i\}$ and $i > 1$, then

$$\oplus_{k \in E_i} b_{m_i k} \ominus (\oplus_{k \in E_i} b_{n_i k}) \geq h_2,$$

or

$$\oplus_{k \in E_i} b_{n_i k} \ominus (\oplus_{k \in E_i} b_{m_i k}) \geq h_2.$$

(vi) For each $H \subseteq \{p_i + 1, \ldots, \}$,

$$\oplus_{k \in H} b_{m_i k} \ominus (\oplus_{k \in H} b_{n_i k}) \leq h_4,$$

or

$$\oplus_{k \in H} b_{n_i k} \ominus (\oplus_{k \in H} b_{m_i k}) \leq h_4.$$

Thus, we can find a $Q \subseteq \mathbf{N}$ and an infinite subset G of $\mathbf{N}$ such that $E_i \subseteq Q$ if $i \in G$ and $E_i \cap Q = \emptyset$ if $i \in \mathbf{N} \backslash G$, and $\{\oplus_{k \in Q} b_{nk}\}_{n \in \mathbf{N}}$ is an order topology τ_0^L-Cauchy sequence.

On the other hand, it follows from (iv), (v), and (vi) and Lemma 2 that for $i \in G$, $i > 1$, we have

$\oplus_{k \in Q} b_{m_i k} \ominus (\oplus_{k \in Q} b_{n_i k}) \geq h_2 \ominus h_3 \ominus h_4$ or $\oplus_{k \in Q} b_{n_i k} \ominus (\oplus_{k \in Q} b_{m_i k}) \geq h_2 \ominus h_3 \ominus h_4$. This contradicts $\{\oplus_{k \in Q} b_{nk}\}_{n \in \mathbf{N}}$ is an order topology τ_0^L-Cauchy sequence and so this conclusion is true.

Thus, the Antosik–Mikusinski theorem (Junde *et al.*, 2003) shows that $\{b_{ii}\}$ is order topology τ_0^L convergent to 0. This contradicts (2) and so the theorem is proved.

The following important conclusion can be obtained from Theorem 1 immediately:

Theorem 2. *Let $(L, \perp, \oplus, 0, 1)$ be a complete totally ordered connect effect algebra, for each $i \in \mathbf{N}$, $\{a_{i\alpha}\}_{\alpha \in \Lambda}$ be an orthogonal set of L. If for each subset $\triangle$ of Λ, the $\oplus$-sum sequence $\{\oplus_{\alpha \in \triangle} a_{i\alpha}\}_{i \in \mathbf{N}}$ is order topology τ_0^L convergent, then $\{a_{i\alpha}\}_{\alpha \in \Lambda}$ are uniformly $\oplus$-summable with respect to $i \in \mathbf{N}$.*

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